Int. J. Solids Structures, 1975, Vol. 11, pp. 961–971. Pergamon Press. Printed in Gt. Britain.

IMPROVED UPPER AND LOWER BOUNDS FOR DEFLECTIONS OF ORTHOTROPIC CANTILEVER BEAMS†

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(*Received* 30 *September* 1974)

Abstract—Formulas for upper and lower deflection bounds, in terms of appropriately applied approximations to potential and complementary energy expressions, are evaluated on the basis of variational problems which involve fourth-order ordinary Euler differential equations, with associated Euler and constraint boundary conditions. The paper obtains new information on the order of magnitude of effects which modify the results of elementary beam theory through the influence of transverse shear and normal strain deformations, including the delineation of boundary layer effects, with one or two such layers, depending on the degree of orthotropy of the material of the beam.

INTRODUCTION

In what follows we extend our earlier work on upper and lower bounds for the deflection of end-loaded laminated cantilever beams of narrow rectangular cross section through use of the principles of minimum potential and maximum complementary energy [1, 2].

Specifically, we propose to improve the bounds

$$\frac{C_{L3}}{C_o} \leq \frac{C}{C_o} \leq \frac{C_{U1}}{C_o}$$

for the flexibility coefficient C of an orthotropic homogeneous beam, where $C_o = a^3/2Ec^3$ is the value of C in accordance with elementary theory and where

$$\frac{C_{U1}}{C_o} = 1 + \frac{6Ec^2}{5Ga^2},$$

and

$$\frac{C_{L3}}{C_o} = 1 - \frac{3\nu^2}{\sqrt{15}}\sqrt{\frac{G}{E_y}}\frac{c}{a}\left(\coth\sqrt{\frac{15E_y}{G}}\frac{a}{c} - \sqrt{\frac{G}{15E_y}}\frac{c}{a}\right) + \frac{Ec^2}{Ga^2},$$

with

$$\frac{C_{L3}}{C_o} = 1 - \frac{3\nu^2}{\sqrt{15}}\sqrt{\frac{G}{E_y}}\frac{c}{a} + \left(\frac{E}{G} + \frac{\nu^2}{5}\frac{G}{E_y}\right)\frac{c^2}{a^2},$$

effectively, when $0 \leq c/a \leq \sqrt{E_y/G}$, in such a way that the value of the factor 6/5 in C_{U1} as well as the occurrence of the negative, linear, c/a-term in C_{L3} are established as quantitatively significant aspects of the behavior of the actual values of C/C_o.

Going beyond this we will show that while the term $(6E/5G)(c^2/a^2)$ originates from what may be designated as the interior solution contribution of the boundary value problem of the end-loaded cantilever, we have that the terms linear in c/a which occur in C_{L3}/C_o, as well as the corresponding terms in the quantities C_{L4}/C_o and C_{U2}/C_o which are obtained in what follows, are in fact boundary layer solution contributions of our boundary value problem. Insofar as these boundary layer solution contributions are concerned we mention in particular our detailed analysis of the structure of the layer, with the number and the magnitudes of distinct characteristic lengths which are found being well-defined functions of certain moduli ratios of the material of the beam.

In regard to the upper and lower bound formulas used in this work, we have previously noted

†Preparation of this paper, which represents an extended version of material included in the doctoral dissertation of the first-named author (UCSD, September 1974) has been supported by the Office of Naval Research, Washington, D.C.

that "inequalities of a similar nature have been stated earlier by others, in particular by C. Weber, but as far as we know not for the problem which is here under consideration." A review of the literature undertaken by us since has established that the above statement should be amplified by referring to a specific publication by Weber[3], which considers in particular an isotropic homogeneous beam on two simple supports with a concentrated load at midspan, in such a way that the formulation for one-half of this beam is effectively equivalent to the formulation of our cantilever beam problem. When interpreted in this manner the work of Weber includes a formula $C_{L2} \leq C \leq C_{U1}$, where $C_{U1}/C_o = 1 + 12(1+\nu)c^2/5a^2$ and $C_{L2}/C_o = 1 - \nu^2 + 2(1+\nu)c^2/a^2$, consistent with the formulas for laminated orthotropic beams given in [1].

FORMULATION

We restate our plane stress problems once more in the form of differential equations

$$\sigma_{x,x} + \tau_{,y} = 0, \quad \tau_{,x} + \sigma_{y,y} = 0, \tag{1}$$

$$u_{,x} = \frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E_m}, \quad v_{,y} = \frac{\sigma_y}{E_y} - \nu\frac{\sigma_x}{E_m}, \quad u_{,y} + v_{,x} = \frac{\tau}{G}, \tag{2}$$

for the domain $-c \leq y \leq c$, $0 \leq x \leq a$, together with boundary conditions of the form

$$x = 0; \quad \sigma_x = 0, \quad v = -V, \tag{3}$$

$$x = a; \quad u = 0, \quad v = 0, \tag{4}$$

$$y = \pm c; \quad \sigma_y = 0, \quad \tau = 0. \tag{5}$$

In this we assume that $E_m = (EE_y)^{1/2}$ and that E, E_y, ν and G are given constants, with $\nu^2 < 1$ as condition for strain energy positive definiteness.

Evidently the uniform end deflection V will be associated with an end force P given by

$$P = \int_{-c}^{c} \tau(0, y)\, dy = \int_{-c}^{c} \tau(x, y)\, dy, \tag{6}$$

in the form

$$V = CP, \tag{7}$$

and our objective is the determination of upper and lower bounds for C, as a function of the given parameters E, E_y, ν, G, c and a.

In order to obtain these bounds we make use of the fact that the work quantity PV is bounded in terms of potential and complementary energy approximations $\mathbf{I}_d$ and $\mathbf{I}_s$,

$$\mathbf{I}_s \leq \frac{1}{2} VP \leq \mathbf{I}_d, \tag{8}$$

where

$$\mathbf{I}_s = VP - \frac{1}{2}\int_{-c}^{c}\int_{o}^{a}\left\{\frac{\boldsymbol{\sigma}_x^{\,2}}{E} - 2\nu\frac{\boldsymbol{\sigma}_x\boldsymbol{\sigma}_y}{E_m} + \frac{\boldsymbol{\sigma}_y^{\,2}}{E_y} + \frac{\boldsymbol{\tau}^2}{G}\right\} dx\, dy, \tag{9}$$

and

$$\mathbf{I}_d = \frac{1}{2}\int_{-c}^{c}\int_{o}^{a}\left\{\frac{E\mathbf{u}_{,x}^2 + 2\nu E_m \mathbf{u}_{,x}\mathbf{v}_{,y} + E_y\mathbf{u}_{,y}^2}{1-\nu^2} + G(\mathbf{u}_{,y} + \mathbf{v}_{,x})^2\right\} dx\, dy. \tag{10}$$

In this $\boldsymbol{\sigma}_x$, $\boldsymbol{\sigma}_y$ and $\boldsymbol{\tau}$ must satisfy the equilibrium differential equations together with the prescribed stress boundary conditions, and $\mathbf{u}$ and $\mathbf{v}$ must be differentiable functions which satisfy the displacement boundary conditions of the given problem. Our procedure from here on is then to minimize $\mathbf{I}_d$ and maximize $\mathbf{I}_s$ with reference to certain systems of functions $\boldsymbol{\sigma}_x$, $\boldsymbol{\sigma}_y$, $\boldsymbol{\tau}$, $\mathbf{u}$ and $\mathbf{v}$, and to discuss the bound relations

$$\frac{C_L}{C_o} \leq \frac{C}{C_o} \leq \frac{C_U}{C_o}, \tag{11}$$

where

$$\frac{C_L}{C_o} = \frac{V^2/2C_o}{\mathbf{I}_{d,\min}}, \quad \frac{C_U}{C_o} = \frac{V^2/2C_o}{\mathbf{I}_{s,\max}}, \tag{12}$$

which follow from (8) upon eliminating P through use of equation (7).

UPPER BOUND CALCULATION

We consider an equilibrium stress system of the form

$$a\boldsymbol{\sigma}_x = VE\rho\Lambda[\eta\xi + (20\eta^3 - 12\eta)F(\xi)], \tag{13}$$

$$a\boldsymbol{\tau} = VE\rho^2\Lambda\left[\frac{1}{2}(1-\eta^2) - (5\eta^4 - 6\eta^2 + 1)F'(\xi)\right], \tag{14}$$

$$a\boldsymbol{\sigma}_y = VE\rho^3\Lambda(\eta^5 - 2\eta^3 + \eta)F''(\xi), \tag{15}$$

where $\xi = x/a$ and $\eta = y/c$ are dimensionless coordinates, $\rho = c/a$, primes indicate differentiation with respect to ξ, and the constant Λ as well as the function $F(\xi)$ are arbitrary except for the condition $F(0) = 0$.

We introduce equations (13–15) into (9) and perform the indicated integrations with respect to η. In this way we obtain an expression for $\mathbf{I}_s$ of the form

$$\mathbf{I}_s = \frac{V^2}{2C_o}\left[\frac{2}{3}\Lambda - \frac{1}{9}\Lambda^2\left(1 + \frac{6}{5}\frac{E}{G}\rho^2 + I\right)\right], \tag{16}$$

where

$$I = \int_0^1 \left[\frac{576}{7}F^2 + \frac{256}{35}\frac{\nu E}{E_m}\rho^2 FF'' + \frac{128}{35}\frac{E}{G}\rho^2(F')^2 + \frac{128}{385}\frac{E}{E_y}\rho^4(F'')^2\right] d\xi$$
$$- \frac{48}{35}\rho^2\left[\left(\frac{E}{G} - \frac{\nu E}{E_m}\right)F(1) + \frac{\nu E}{E_m}F'(1)\right]. \tag{17}$$

Maximization of $\mathbf{I}_s$ with respect to Λ, for fixed F, gives as expression for Λ,

$$\Lambda = \frac{3}{1 + (6E/5G)\rho^2 + I}, \tag{18}$$

and therewith

$$\mathbf{I}_{s,\max} = \frac{V^2}{2C_o}\frac{1}{1 + (6E/5G)\rho^2 + I}. \tag{19}$$

Introduction of (19) into equation (12) gives

$$\frac{C_U}{C_o} = 1 + \frac{6}{5}\frac{E}{G}\rho^2 + I, \tag{20}$$

where it remains to choose the function $F(\xi)$, with the simplest choice, $F(\xi) = 0$, evidently leading to the previously obtained bound

$$C_{U1}/C_o = 1 + (6E/5G)\rho^2.$$

In what follows we will obtain a value $I_{\min}$ by determining the function $F(\xi)$ from the variational equation $\delta I = 0$, together with the constraint condition $F(0) = 0$.

Application of the standard rules of the calculus of variations now leads to the Euler differential equation

$$\frac{E}{E_y}F'''' - \frac{11}{\rho^2}\left(\frac{E}{G} - 2\frac{\nu E}{E_m}\right)F'' + \frac{495}{2\rho^4}F = 0 \tag{21}$$

and to the Euler boundary conditions

$$\begin{aligned} &\frac{E}{E_y}F''(1) + \frac{11}{\rho^2}\frac{\nu E}{E_m}F(1) = \frac{33}{16\rho^2}\frac{\nu E}{E_m}, \quad F''(0) = 0, \\ &\frac{E}{E_y}F'''(1) - \frac{11}{\rho^2}\left(\frac{E}{G} - \frac{\nu E}{E_m}\right)F'(1) = \frac{33}{16\rho^2}\left(\frac{E}{G} - \frac{\nu E}{E_m}\right). \end{aligned} \tag{22}$$

Appropriate integrations by parts in the expression for I, in conjunction with equations (21) and (22), now permit a simplification of the expression for $I_{\min}$, as follows:

$$I_{\min} = -\frac{24}{5}\rho^2\left\{\frac{E}{G}F(1) + \frac{\nu E}{E_m}[F'(1) - F(1)]\right\}. \tag{23}$$

It remains to solve the differential equation for F, subject to the associated four boundary conditions, and to introduce the value of $I_{\min}$ into equation (20) for C_U/C_o. In this way we obtain an expression for C_{U2}/C_o, which can be written in the form

$$\frac{C_{U2}}{C_o} = 1 + \frac{6}{5}\lambda^2 - \frac{99}{70\chi}\left\{\nu^2\chi_1 + 2\nu\left(\frac{\lambda^2}{\sigma^2} - \nu\right)\chi_2 + \left(\frac{\lambda^2}{\sigma^2} - \nu\right)^2\chi_3\right\}, \tag{24}$$

where $\lambda = \rho\sqrt{(E/G)}$, $\sigma = \rho\sqrt[4]{(E/E_y)}$, and

$$\begin{aligned} \chi_1 &= K_1K_2(K_1^2 - K_2^2), \\ \chi_2 &= (K_1^2 + 11\nu/\sigma^2)K_2 \tanh K_1 - (K_2^2 + 11\nu/\sigma^2)K_1 \tanh K_2, \\ \chi_3 &= (K_1^2 - K_2^2)\,\mathrm{tank}\, K_1 \tanh K_2, \\ \chi &= (K_1^2 + 11\nu/\sigma^2)^2 K_2 \tanh K_1 - (K_2^2 + 11\nu/\sigma^2)^2 K_1 \tanh K_2, \end{aligned} \tag{25}$$

with K_1 and K_2 being the two roots with positive real parts of the characteristic equation

$$\sigma^4K^4 - 11(\lambda^2 - 2\nu\sigma^2)K^2 + 495/2 = 0, \tag{26}$$

that is,

$$\begin{Bmatrix} K_1 \\ K_2 \end{Bmatrix} = \frac{1}{\sigma}\sqrt{\left(\frac{11}{2}\right)}\left\{\left(\frac{\lambda^2}{\sigma^2} - 2\nu\right) \pm \left[\left(\frac{\lambda^2}{\sigma^2} - 2\nu\right)^2 - \frac{90}{11}\right]^{1/2}\right\}^{1/2}, \tag{27}$$

when $[2\nu + \sqrt{(90/11)}]^{1/2}\sigma < \lambda$, and

$$\begin{Bmatrix} K_1 \\ K_2 \end{Bmatrix} = \frac{1}{\sigma}\sqrt{\left(\frac{11}{4}\right)}\left\{\left[\frac{\lambda^2}{\sigma^2} + \left(\sqrt{\left(\frac{90}{11}\right)} - 2\nu\right)\right]^{1/2} \pm i\left[\left(\sqrt{\left(\frac{90}{11}\right)} + 2\nu\right) - \frac{\lambda^2}{\sigma^2}\right]^{1/2}\right\} \tag{28}$$

when $\lambda < [2\nu + \sqrt{(90/11)}]^{1/2}\sigma$.

LOWER BOUND CALCULATION

We assume as expressions for displacements

$$\mathbf{v} = V\left\{f_o(\xi) + \left(\eta^2 - \frac{1}{3}\right)f_2(\xi)\right\}, \quad \mathbf{u} = V\rho\left\{\eta g_1(\xi) + \frac{1}{3}(\eta^3 - \eta)g_3(\xi)\right\}, \tag{29}$$

where

$$f_o(0) = -1, \quad f_2(0) = 0, \quad f_o(1) = f_2(1) = g_1(1) = g_3(1) = 0. \tag{30}$$

We introduce (29) into equation (10) and carry out all η-integrations. This gives

$$\mathbf{I}_d = \frac{V^2}{2C_o}\int_0^1 \left\{\frac{1}{1-\nu^2}\left[\frac{1}{3}(g_1')^2 - \frac{4}{45}g_1'g_3' + \frac{8}{945}(g_3')^2 + \frac{4\nu}{3\sigma^2}f_2\left(g_1' - \frac{2}{15}g_3'\right) + \frac{4}{3\sigma^4}f_2^2\right]\right.$$
$$\left. + \frac{1}{\lambda^2}\left[(g_1 + f_o')^2 + \frac{4}{45}(g_3 + f_2')^2\right]\right\} \mathrm{d}\xi. \tag{31}$$

We now set $\delta \mathbf{I}_d = 0$ and obtain the Euler differential equations

$$\left(g_1 - \frac{2}{15}g_3\right)'' + \frac{2\nu}{\sigma^2}f_2' - 3\frac{1-\nu^2}{\lambda^2}(g_1 + f_o') = 0, \tag{32}$$

$$(g_1 + f_o')' = 0, \tag{33}$$

$$\left(\frac{4}{21}g_3 - g_1\right)'' - \frac{2\nu}{\sigma^2}f_2' - 2\frac{1-\nu^2}{\lambda^2}(g_3 + f_2') = 0, \tag{34}$$

$$\frac{1-\nu^2}{15\lambda^2}(g_3 + f_2')' - \frac{1}{\sigma^4}f_2 - \frac{\nu}{2\sigma^2}\left(g_1 - \frac{2}{15}g_3\right)' = 0, \tag{35}$$

as well as the Euler boundary conditions

$$g_1'(0) = g_3'(0) = 0. \tag{36}$$

Integration by parts in (31), with the use of equations (30) and (32) to (36), gives as expression for $\mathbf{I}_{d,\min}$,

$$\mathbf{I}_{d,\min} = \frac{V^2}{2C_o}\frac{g_1(0) + f_o'(0)}{\lambda^2}. \tag{37}$$

With this there follows from equation (12) as expression for the new lower bound,

$$\frac{C_{L4}}{C_o} = \frac{\lambda^2}{g_1(0) + f_o'(0)}. \tag{38}$$

In order to evaluate (38), we must solve the system (32) to (35) subject to the boundary conditions (30) and (36). The result can be written in the form

$$\frac{C_{L4}}{C_o} = 1 + \frac{6}{5}\lambda^2 - \frac{3}{x}\left\{\nu^2 x_1 + 2\nu\left(\frac{\lambda^2}{\sigma^2} - \nu\right)x_2 + \left(\frac{\lambda^2}{\sigma^2} - \nu\right)^2 x_3\right\}, \tag{39}$$

where

$$\begin{aligned} x_1 &= k_1k_2(k_1^2 - k_2^2), \\ x_2 &= k_1k_2(k_1 \tanh k_1 - k_2 \tanh k_2), \\ x_3 &= (k_1^2 - k_2^2)\tanh k_1 \tanh k_2, \\ x &= k_1k_2(k_1^3 \tanh k_1 - k_2^3 \tanh k_2), \end{aligned} \tag{40}$$

with k_1, k_2 being the two roots with positive real parts of the characteristic equation

$$\sigma^4k^4 - 15\lambda^2k^2 + 525(1-\nu^2) = 0, \tag{41}$$

that is

$$\begin{Bmatrix} k_1 \\ k_2 \end{Bmatrix} = \frac{1}{\sigma}\sqrt{\left(\frac{15}{2}\right)}\left\{\frac{\lambda^2}{\sigma^2} \pm \left[\frac{\lambda^4}{\sigma^4} - \frac{28(1-\nu^2)}{3}\right]^{1/2}\right\}^{1/2} \tag{42}$$

when $[28(1-\nu^2)/3]^{1/4}\sigma < \lambda$, and

$$\begin{Bmatrix} k_1 \\ k_2 \end{Bmatrix} = \frac{1}{\sigma}\sqrt{\left(\frac{15}{4}\right)}\left\{\left[\sqrt{\left(\frac{28(1-\nu^2)}{3}\right)} + \frac{\lambda^2}{\sigma^2}\right]^{1/2} \pm i\left[\sqrt{\left(\frac{28(1-\nu^2)}{3}\right)} - \frac{\lambda^2}{\sigma^2}\right]^{1/2}\right\} \tag{43}$$

when $\lambda < [28(1-\nu^2)/3]^{1/4}\sigma$.

ASYMPTOTIC BOUND FORMULAS

Consideration of the expressions (27), (28), (42), and (43) for the roots K_i and k_i of the characteristic equations (26) and (41) indicates the existence of the following order of magnitude relations,

$$\text{(i)}\ \frac{\lambda^2}{\sigma^2} = O(1):\quad \operatorname{Re}(K_i, k_i) = O\left(\frac{4}{\sigma}\right), \tag{44a}$$

$$\text{(ii)}\ 1 \ll \frac{\lambda^2}{\sigma^2}:\quad K_1, k_1 = O\left(\frac{4}{\sigma}\frac{\lambda}{\sigma}\right),\quad K_2, k_2 = O\left(\frac{4}{\sigma}\frac{\sigma}{\lambda}\right). \tag{44b}$$

Furthermore, the hyperbolic tangent functions in (25) and (40) can be effectively replaced by unity for sufficiently large real parts of their arguments, say when

$$4 \le \operatorname{Re}\{K_i, k_i\}, \tag{45}$$

and the bound formulas (24) and (39) then can be written in the form

$$\frac{C_{U2}}{C_o} = 1 + \frac{6}{5}\lambda^2 - C_1\sigma - C_2\sigma^2 - C_3\sigma^3, \tag{46}$$

$$\frac{C_{L4}}{C_o} = 1 + \frac{6}{5}\lambda^2 - c_1\sigma - c_2\sigma^2 - c_3\sigma^3, \tag{47}$$

where,

$$C_1 = \frac{9\sqrt{(11)}\nu^2}{70}\frac{[\lambda^2 + (\sqrt{(90/11)} - 2\nu)\sigma^2]^{1/2}\sigma}{\lambda^2 + (\sqrt{(45/22)} - \sqrt{(22/45)}\nu^2)\sigma^2}, \tag{48}$$

$$c_1 = \frac{3\nu^2[\lambda^2 + \sqrt{(28(1-\nu^2)/3)}\sigma^2]^{1/2}\sigma}{15\quad \lambda^2 + \sqrt{(7(1-\nu^2)/3)}\sigma^2}, \tag{49}$$

$$C_2 = \frac{9\nu}{35}\frac{(1 - \sqrt{(22/45)}\nu)(\lambda^2 - \nu\sigma^2)}{\lambda^2 + (\sqrt{(45/22)} - \sqrt{(22/45)}\nu^2)\sigma^2}, \tag{50}$$

$$c_2 = \frac{2\nu}{5}\frac{\lambda^2 - \nu\sigma^2}{\lambda^2 + \sqrt{(7(1-\nu^2)/3)}\sigma^2}, \tag{51}$$

$$C_3 = \frac{3}{35\sqrt{(10)}}\frac{(\lambda^2 - \nu\sigma^2)^2[\lambda^2 + (\sqrt{(90/11)} - 2\nu)\sigma^2]^{1/2}}{[\lambda^2 + (\sqrt{(45/22)} - \sqrt{(22/45)}\nu^2)\sigma^2]\sigma^3}, \tag{52}$$

$$c_3 = \frac{1}{5\sqrt{(35(1-\nu^2))}}\frac{(\lambda^2 - \nu\sigma^2)^2[\lambda^2 + \sqrt{(28(1-\nu^2)/3)}\sigma^2]^{1/2}}{[\lambda^2 + \sqrt{(7(1-\nu^2)/3)}\sigma^2]\sigma^3}. \tag{53}$$

We note that with σ and λ both proportional to ρ we have that C_i as well as c_i are independent of ρ so that C_{U2}/C_o as well as C_{L4}/C_o come out to be third-degree polynomials in ρ.

Replacement of the hyperbolic tangent functions by unity is readily seen to be equivalent to the assumption that the functions F, f_i and g_i describe a boundary layer phenomenon in the region adjacent to $\xi = 1$. The form of equations (44) and (45) gives as conditions for the existence of a boundary layer the relations

$$\frac{\lambda^2}{\sigma^2} = O(1): \quad \sigma \lessdot 1, \tag{54a}$$

$$1 \ll \frac{\lambda^2}{\sigma^2}: \quad \lambda \lessdot 1. \tag{54b}$$

In view of the defining relations for λ and σ, these conditions may be written in the alternate form

$$\frac{E}{G}\sqrt{\frac{E_y}{E}} = O(1): \quad \rho \lessdot \sqrt[4]{\frac{E_y}{E}} \tag{55a}$$

$$1 \ll \frac{E}{G}\sqrt{\frac{E_y}{E}}: \quad \rho \lessdot \sqrt{\frac{G}{E}}. \tag{55b}$$

We next use equation (44) to obtain information concerning the width of the boundary layers which are associated with the solution of the given problem. It is apparent that there exist either one or two boundary layers, in accordance with the following pattern

$$\frac{E}{G}\sqrt{\frac{E_y}{E}} = O(1): \quad \frac{b}{c} = O\left(\sqrt[4]{\frac{E_y}{E}}\right) \tag{56a}$$

$$1 \ll \frac{E}{G}\sqrt{\frac{E_y}{E}}: \quad \frac{b_1}{c} = O\left(\sqrt{\frac{G}{E}}\right), \quad \frac{b_2}{c} = O\left(\sqrt{\frac{E_y}{G}}\right). \tag{56b}$$

As $(E/G)\sqrt{(E_y/E)}$ approaches unity from above the two layers of width b_1 and b_2 coalesce into the one layer of width b.

ASYMPTOTIC FORMULAS FOR ISOTROPIC BEAMS

Setting $E_y = E$ and $E = 2(1+\nu)G$, we will have $\sigma = \rho$, and $\lambda = \rho\sqrt{2(1+\nu)}$ and, in accordance with equation (54a), the asymptotic formulas (46) to (53) now apply as long as $\rho \lessdot 1$.

We will refrain from rewriting equation (46) to (53) for this special case and instead refer to Fig.

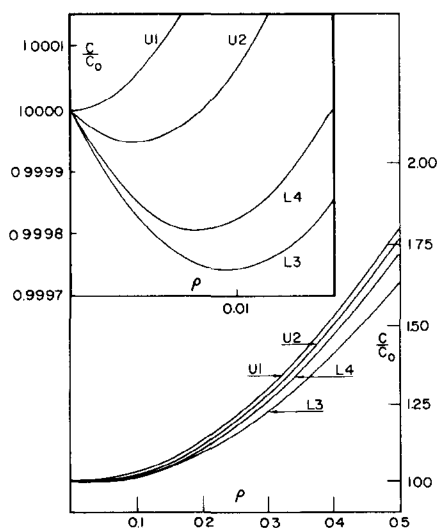

Fig. 1. Dimensionless flexibilities C_{U1}/C_o, C_{U2}/C_o, C_{L3}/C_o and C_{L4}/C_o as functions of $\rho = c/a$ for isotropic beams with $\nu = 1/3$.

Table 1. Upper and lower bounds for influence coefficients for isotropic beams with $\nu = 1/3$

ρ	C_{L4}/C_o	C_{U2}/C_o
0.0	1.0000	1.0000
0.1	1.0261	1.0290
0.2	1.1142	1.1208
0.3	1.2633	1.2749
0.4	1.4729	1.4908
0.5	1.7423	1.7680
0.6	2.0709	2.1061
0.7	2.4579	2.5046
0.8	2.9027	2.9630
0.9	3.4046	3.4809
1.0	3.9629	4.0577

1 which shows the values of C_{U2}/C_o and C_{L4}/C_o for $\nu = 1/3$ in the range $0 \leq \rho < 1/2$ together with the previously obtained less accurate bounds. The curves in the inset show the behavior of the bounds in the neighborhood of $\rho = 0$. We see that the effect of the "dominant" linear terms in ρ, which make $C/C_o < 1$ in a small neighborhood of $\rho = 0$ is in fact of no practical significance whatsoever for the case of the isotropic beam.

We supplement the results shown in Fig. 1 by a short table which gives numerical values in the range $0 \leq \rho \leq 1{\cdot}0$, and which shows the smallness of the error associated with the approximation $C \approx \frac{1}{2}\left(C_{U2} + C_{L4}\right)$, up to values of the depth-span ratio for which it is no longer appropriate to use the word "beam."

RESULTS FOR ORTHOTROPIC BEAMS

Figure 2 shows values of the bounds as a function of λ for several values of the auxiliary parameter $\mu = \sigma^2/\lambda^2$, on the basis of the exact bound expressions (24) and (39), as well as on the basis of the asymptotic expressions (46) and (47). We see that the exact and the asymptotic values coincide effectively in the entire range of λ-values shown, for small values of μ. As the values of μ increase, the range of validity of the asymptotic formula extends over a smaller and smaller λ-range. Specifically, if we write $\sigma/\lambda = \sqrt{\mu}$ and note that according to (54a) we must now have

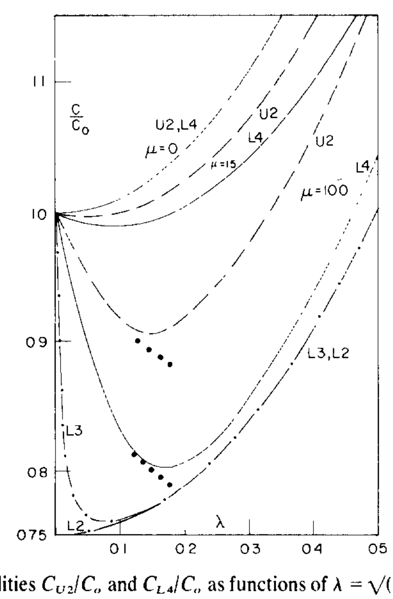

Fig. 2. Dimensionless flexibilities C_{U2}/C_o and C_{L4}/C_o as functions of $\lambda = \sqrt{(E/G)}\,c/a$ for orthotropic beams with $\nu = 1/2$ and $\mu = G/\sqrt{(EE_y)} = 0, 1{\cdot}5, 100$, together with values of C_{L3}/C_o and C_{L2}/C_o for $\mu = 100$.

$\sigma \leqslant 1$ we find that we must also have $\lambda\sqrt{\mu} \leqslant 1$, or $\lambda \leqslant 1/\sqrt{\mu}$, as condition for the validity of the asymptotic formula. For $\mu = 100$ this means that the asymptotic results should deviate from the exact bound results as soon as $0{\cdot}1 \leqslant \lambda$, a conclusion which is in fact substantiated by the dots in Fig. 2.

Our calculations also show that the effect of the prevented end-section transverse normal strain, which is insignificant for isotropic beams, can be quite significant for beams with strong orthotropy.

We supplement our discussion with an explicit statement of the exact *bound* results (24) and (39), in comparison with results of exact *solutions* of the given boundary value problem, for the limiting cases $E_y = 0$ and $E_y = \infty$.

We deduced in [2] that when $E_y = 0$, then

$$\frac{C}{C_o} = \frac{\lambda^2}{1 - \dfrac{\tanh \lambda\sqrt{(3/(1-\nu^2))}}{\lambda\sqrt{(3/(1-\nu^2))}}} \approx 1 - \nu^2 + \frac{6}{5}\lambda^2 - \frac{3}{1-\nu^2}\frac{\lambda^4}{175} \pm \cdots. \tag{57a}$$

We now find from equation (24) and (39),

$$\frac{C_{U2}}{C_o} = 1 - \frac{33}{70}\nu^2 + \frac{6}{5}\lambda^2 \frac{(\nu^2 - 3\lambda^2/11)^2}{1 - 22\nu^2/45 + 2\lambda^2/15}, \tag{57b}$$

$$\frac{C_{L4}}{C_o} = 1 - \nu^2 + \frac{6}{5}\lambda^2 - \frac{3}{1-\nu^2}\frac{\lambda^4/175}{1 + 3\lambda^2/35(1-\nu^2)}. \tag{57c}$$

Numerical calculations for $\nu = 1/2$ show that now C_{L4} is closer to C than is C_{U2}. As shown in Table 2, we have that when $0 \leq \lambda \leq 0{\cdot}5$ C_{L4} agrees with C to within 1/100 of one percent. We also note that C_{U2} as well as C_{L4} are discontinuous at $\lambda = 0$, just as C comes out to be, but that the magnitude of the discontinuity of C_{U2} does not agree with that of C, whereas there is agreement for C_{L4} and, incidentally, also for C_{L3}, as may be readily deduced from the formula on page 961 which leads to the result $C_{L3}/C = 1 - \nu^2 + \lambda^2 + O(\lambda^4)$ for the limiting case $E_y = 0$.

When $E_y = \infty$ the bound formulas (24) and (39) reduce to the following form

$$\frac{C_{U2}}{C_o} = 1 + \frac{6}{5}\lambda^2 - \frac{3\tanh 3\sqrt{(5/2)}/\lambda}{70\sqrt{(5/2)}}\lambda^3, \tag{58a}$$

$$\frac{C_{L4}}{C_o} = 1 + \frac{6}{5}\lambda^2 - \frac{\tanh\sqrt{(35(1-\nu^2)}/\lambda}{5\sqrt{(35(1-\nu^2))}}\lambda^3. \tag{58b}$$

Table 2. Comparison of bounds with the exact flexibility coefficient when $E_y = 0$ and $\nu = 1/2$

λ	C/C_o	C_{L4}/C_o	C_{U2}/C_o
0.0	0.7500	0.7500	0.8657
0.1	0.7619	0.7619	0.8782
0.2	0.7979	0.7979	0.9152
0.3	0.8578	0.8578	0.9770
0.4	0.9414	0.9414	1.0633
0.5	1.0486	1.0486	1.1738
0.6	1.1792	1.1791	1.3085
0.7	1.3329	1.3328	1.4669
0.8	1.5096	1.5092	1.6488
0.9	1.7089	1.7082	1.8540
1.0	1.9306	1.9294	2.0821

The exact solution for this case can be obtained by extending results in [4], as shown in the Appendix. We find

$$\frac{C}{C_o} = 1 + \frac{6}{5}\lambda^2 - 2\sum_{n=1}^{\infty} \frac{\tanh k_n/\lambda}{(k_n/\lambda)^2}, \tag{58c}$$

with k_n as the sequence of positive roots of the transcendental equation $\tan k_n = k_n$.

A comparison of the bound results (58a) and (58b) with the exact result (58c) shows that the bounds obtained here are in agreement with the exact formula insofar as terms up to the order ρ^2 are concerned. For small λ, the remaining terms in equations (58a, b, c) are of the order λ^3. Numerical results obtained using (58a) and (58b) when $\nu = 1/2$ are given in Table 3. In view of the closeness of these bounds a calculation of the exact expression (58c) which involves summation of an infinite series, is not undertaken.

Table 3. Upper and lower bounds for flexibility coefficients for orthotropic beams when $E_y = \infty$ and $\nu = 1/2$

λ	C_{L4}/C_o	C_{U2}/C_o
0.0	1.0000	1.0000
0.1	1.01196	1.01197
0.2	1.0476	1.0478
0.3	1.1069	1.1073
0.4	1.1895	1.1903
0.5	1.2951	1.2967

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APPENDIX

In what follows we indicate the derivation of the exact value (58c) of the flexibility coefficient C for the limiting case $E_y = \infty$, in extension of work by Hildebrand[4],

We have from [4] as expressions for σ_x and τ,

$$\sigma_x = \frac{3P}{2\rho c}\left\{\xi\eta + \frac{2}{3}\sum_{n=1}^{\infty} \frac{\lambda \sin k_n\eta}{k_n \sin k_n} \frac{\sinh k_n\xi/\lambda}{\cosh k_n/\lambda}\right\}, \tag{A1}$$

$$\tau = \frac{3P}{4c}\left\{1 - \eta^2 + \frac{4}{3}\sum_{n=1}^{\infty} \frac{\cos k_n\eta - \cos k_n}{k_n \sin k_n} \frac{\cosh k_n\xi/\lambda}{\cosh k_n/\lambda}\right\}, \tag{A2}$$

with k_n as the sequence of positive roots of $\tan k_n = k_n$.

We also have from [4] as equations for $v(\xi, \eta) = v(\xi)$,

$$\frac{v''(\xi)}{a} = \frac{\tau_{,\xi}}{G} - \frac{\sigma_{x,\eta}}{E\rho}, \quad v(1) = 0, \quad v'(1) = \frac{a}{G}\int_0^1 \tau(1, \eta)\, d\eta. \tag{A3}$$

We solve (A3) for $v(\xi)$ and in this way find

$$V = -v(0) = \frac{P}{2\rho G}\left\{1 + \frac{1}{\lambda^2} + 2\sum_1^{\infty} \frac{1}{k_n^2}\left(1 - \frac{\lambda}{k_n}\right)\tanh\frac{k_n}{\lambda}\right\}, \tag{A4}$$

and from this, with $C_o = 1/2E\rho^3$,

$$\frac{C}{C_o} = 1 + \lambda^2\left(1 + \sum_1^\infty \frac{2}{k_n^2}\right) - 2\sum_1^\infty \frac{\tanh k_n/\lambda}{(k_n/\lambda)^3}. \tag{A5}$$

Equation (A5) assumes the form (58c) upon establishing, by contour integration, that $\Sigma\, 1/k_n^2 = 1/10$.